On the revision of preferences and choice functions

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Abstract
Preferences and choice functions are subject to be modified when new information is brought to the knowledge of an agent: such or such strategy may suddenly appear more reliable than such other, such or such restaurant has to be retrieved from the list of the ten best tables of the town. We propose here an easy way to perform this revision through a simple modification of the logical chain attached to the agent’s behavior or of the subset chain attached to the choice function: it can be shown indeed that, in the rational case, these chains offer an adequate representation of preferences and of choice functions. Thus the revision problem boils down to adding, retracting or modifying some of the links of the original chain, a perspective that enables an effective treatment of the problem of iterated revision.

1 Introduction
We showed recently [Freu98a, 98b and 99] that it was possible to study the behavior of an agent, be it expressed by a set of preferences or by an inference relation, through some auxiliary subset of the underlying propositional language $\mathcal{L}$. These results easily translate into a pure set-theoretic framework, showing that when the behavior to be studied is of rational type, it can be very simply represented by means of a chain of sets embedded in the sample
space $\Omega$. This means for instance that if $\succ$ is the strict partial order over $\wp(\Omega)$ that represents the preferences of the agent, there exists a sequence of subsets of $\Omega$, $D_0 \subset D_1 \subset \ldots \subset D_n$ such that the preference $A \succ B$ holds if and only if there exits an index $i$ such that $D_i$ intersects $A$ and does not intersect $B$. The behavior of the agent, which can be represented by the set of his preferences, is thus entirely determined by this subjacent chain. This provides a powerful tool to investigate any rational-type behavior, and we showed for instance [Freu.98b] how it is possible to use this kind of representation to extend any partial information on the agent’s preferences.

Representing an agent’s preferences by chains of subsets also reveals itself to be useful when one has to revise or to update some obsolete information relative to these preferences. It may indeed be the case that a given preference $A \succ B$ that does not model accurately enough the behavior of the agent should be retracted from the set of preferences of this agent, or even that it should be replaced by $B \succ A$. To do so, one has to be aware of all the implications this change may induce on the other preferences. To take a trivial example, the preference $A \succ B$ may come as a result from $A \succ C$ and $C \succ B$, so it will be necessary to either reverse one, at least, of these inequalities, or decide to retract both of them, a choice that may of course interfere with other preferences of the initial set. As we shall see, the right tool to handle this kind of problem is provided by chains: the whole revision process boils down to a suitable action on the subjacent chain and this framework leads to a solution that is by far, the easiest to perform, as well as the most accurate one.

A similar work may be undertaken concerning the theory of social choice. The choice functions that are classically associated with rational behaviors are those that satisfy the well-known Arrow properties. It turns out that, as in the case of rational orders, such a choice function is again determined by a chain of embedded subsets. This fact is by no means surprising, as there exists a duality between rational choice functions and rational orders (see for example [Leh.98]). We provide a direct and simple proof of this result, showing how to construct the chain associated with a given rational choice function, and we show, conversely, how a chain of that type induces a rational choice function.

Again, one of the interests of representing rational choice functions by means of chains is that it enables one quite easily to address the problem of correcting some obsolete information conveyed by his functions. Consider for instance the following simple example: you decide to explore all the regions
of France in order to list the best restaurants not only in the whole country, but also town by town, department by department and region by region. At the moment your guide is under press, you learn that the restaurant Gouffé, at Saint Jean de la Ruelle, changed his chef and does not match anymore the best tables of the department. How are you then going to up-date your lists? If these were established by a coherent notation, each restaurant being given a grade, and if your information about Gouffé is that it lost, say, one point, there is no problem in rewriting and reordering all the lists where Gouffé was mentioned. But what if the lists were made without any grade attribution, each geographical region providing the set of its best restaurants? You cannot just remove Gouffé from all the lists where it figured since, still, it might be is a fairly good address, one of the best anyway of Saint Jean de la Ruelle itself and, perhaps, of its immediate neighborhoods. Thus, you would be much satisfied with any rational revising process, as long as your modified lists are still coherent and do not differ too much from the original ones. This, as we will show, is possible through the chain associated with the given choice function.

This paper deals with the general problem of revising an agent’s behavior and will be therefore divided into two parts depending on whether this behavior is represented by a set of preferences or by a choice function. First, we shall work in the framework of preference orders. We shall provide the translation in the set-theoretic context of the logical chain attached to such an order, which we proposed in [Freu.98b], and we shall put in evidence the duality between preference orders and chains. This will enable us to correctly handle the problem of revising these orders: working through the associated chain, we will find a simple solution that furthermore happens to be an optimal one, in a sense that will be made precise. We shall then expose the problem of iterated revision: is it possible to use this procedure to revise a preference set by more than one single preference? We shall show how to apply the chain representation to this problem, and determine when this problem has a solution. In the second part of this work, we shall consider choice functions, focusing our attention on those functions that satisfy the Arrow properties. Again, we shall see that one can associate with any such function a chain of embedded subsets, and that there exists a perfect duality between choice functions and chains of subsets. It will be then easy to treat the problem of choice functions revision, which will boil down to that of a simple set-theoretical surgery.

Throughout this paper, we shall denote by Ω a universal finite set, which
can be considered as a sample set or as a set of basic alternatives, and we shall work on the power set $\mathcal{S} = \mathcal{P}(\Omega)$. Elements of $\mathcal{S}$ will be denoted by capital letters. They may be indifferently interpreted as choices, events, strategies or menus.

Part I

The syntax of preferences

2 Rational Preferences

2.1 Definitions and elementary properties

2.1.1 The notion of rational preference

The central notion of this paper is that of preference. As this term is fairly vague and ambiguous, we must emphasize that, in the present work, we consider only those preferences that suppose the possible giving up of an item in favor of the preferred one: when we say that an agent prefers A to B, we mean that, in the case the agent has to choose between A and B, not only would he take A but he would also reject B. Thus the relation of preference only holds between two items between which one does not really hesitate, like in the "Your money or your life" context. In this situation, it is indeed the case that giving one’s money is preferred, in the above sense, to giving one’s life (and similarly, keeping one’s life is preferred to keeping one’s money). On the contrary, we cannot say that one could prefer tea with milk to tea, since it is meaningless to give up drinking tea in order to drink tea, be it with milk! Rather, we will say that one prefers tea with milk to tea without milk. To make this notion of preference more precise, we will suppose that it is expressed through a strict partial order $\succ$ defined on the power set $\mathcal{S}$ of a (finite) universal set of options $\Omega$. Thus a relation of the type $A \succ B$ is to be interpreted as "in the agent’s mind, A is preferable to B": for this agent, the utility of the item proposed by A is greater than that of the item represented by B. It may be also helpful to keep in mind some other interpretations of the relation $\succ$: thus in a different context we may as well translate $A \succ B$ by "in the agent’s mind the event A is more likely to happen (more plausible) than the event B"; another illustration may also be
"in the agent’s mind, the belief that $A$ is true is more entrenched than the belief that $B$ is true". It mainly in this latter context that in this order was introduced, in a slightly different form, by P. Gärdenfors and D. Makinson in [Gär94]

The above considerations show that the preference relations we consider will satisfy some well-defined properties. Clearly, these relations are expected to be *irreflexive*, as one cannot give up to an item $A$ in order to get it. What is more important, they have to behave properly with respect to *inclusion*: indeed, if menu $A$ is preferred to menu $B$, that is, if one is ready to give up menu $B$ in order to get $A$, then any superset any menu containing $A$ must also be preferred to $B$. In the "Your money or your life" example, this becomes particularly clear if one remembers that larger sets correspond to weaker events. We emphasize here again that we are not dealing here with preferences usually referred to in every-day life, when one has to choose say between two different models of cars: there, it may happen that model $A$ is *altogether* preferred to model $B$ in spite of the fact that $B$ has some qualities that $A$ lacks. The relation between the consequence relation and the order $\succ$ may better be understood if we think of it as a way to compare beliefs or possibilities: if event $A$ occurs more often than event $B$, then the same is true for any event $C$ implied by $A$. Similarly, if my belief in $A$ is more entrenched than my belief in $B$, then so is my belief in any consequence of $A$.

With this in mind, we can now proceed to the definition of a *rational preference order*. For simplicity, we shall make use of the notation $\succeq$ which will be defined as usual by $B \succeq A$ iff one does not have $A \succ B$. We will say that a binary relation $\succ$ over the power set $\mathcal{S} = \wp(\Omega)$ is a relation of *rational preference* if it satisfies the five following properties

$Pr_0$ $A \succ \emptyset$ for all subsets $A$

$Pr_1$ if $A \succ B$ then $A \succeq B$ for all non-empty subsets $A$ and $B$

$Pr_2$ if $A \succ C$ and $A \subseteq B$ then $B \succ C$

$Pr_3$ if $A \cup B \succ B$, then $A \succ B$

$Pr_4$ if $C \succ A$ and $A \succeq B$ then $C \succ B$. 


The first rule states that anything is preferred to the empty menu. The second one expresses asymmetry, (and hence irreflexivity) for nonempty sets: if $A$ is preferred to $B$, then $B$ cannot be preferred to $A$. This translates the fact that, in this work, only strong preferences will be considered. Note that $Pr_1$ is equivalent to the property of connectedness of the relation $\succeq$: given two subsets $A$ and $B$, one has either $A \succeq B$ or $B \succeq A$.

The rule $Pr_2$ was already examined. It expresses the fact that one cannot prefer an event $A$ to an event $C$ unless any single consequence of $A$ is itself preferred to $C$. As we noticed, this rule is a very strong one and shows that the notion of preference that is captured by the relation $\succ$ has little to do with the weak notion of preference we use in every-day life.

The rule $Pr_3$ means that if the set $A \cup B$ is preferred to the set $B$, so that one may give up $B$ in order to get $A \cup B$, then it must be the case that $A$ is itself preferred to $B$. In the context where $\succ$ compares the plausibility of two events, this rule means that if $A \cup B$ is more likely to happen than $B$, then $A$ alone is more likely to happen than $B$.

Finally, the modularity rule $Pr_4$ expresses the natural fact that if $C$ is preferred to $A$ while $B$ is not preferred to $A$, then it must be the case that $C$ is preferred to $B$: indeed if one has a marked preference for bananas rather than oranges, but one does not appreciate apples more than oranges, then one should prefer a banana to an apple. Note that this rule is equivalent to the transitivity of the relation $\succeq$.

It is clear that any rational preference relation $\succ$ is transitive, and therefore a strict partial order, when restricted to the set $\mathcal{S} - \{\emptyset\}$. For short, we will refer to $\succ$ as a rational order on $\mathcal{S}$, although, strictly speaking, it is not an order. Let us give a simple example of such a rational order: we denote by $\Delta$ a fixed subset of $\Omega$, which we may consider as the set of its best elements, and we set, for any subsets $A$ and $B$ of $\Omega$: $A \succ\Delta B$ iff $A \cap \Delta \neq \emptyset$ and $B \cap \Delta = \emptyset$. The subset $A$ is therefore preferred to the subset $B$ iff some of the best elements of $\Omega$ are members of $A$ whilst none of them is a member of $B$. It is straightforward to check that properties $Pr_0$ to $Pr_4$ are satisfied, so that $\succ_{\Delta}$ is a rational order.

We close this section mentioning five derived rules that are satisfied by the restriction of a rational order $\succ$ to the set $\mathcal{S} - \{\emptyset\}$ (here $\overline{X}$ denotes the complementary set of $X$):

1. $A \succ (B \cup C)$ iff $A \succ B$ and $A \succ C$
2. $(B \cup C) \succ A$ iff $B \succ A$ or $C \succ A$
3. \(A \succ (A \cap B)\) iff \(A \cap B \succ A \cap B\)

4. \(\Omega \succ A\) iff \(\overline{A} \succ A\).

5. If \(A \subseteq B\) then \(B \succeq A\)

The first rule may be translated as: a menu \(A\) is preferred to a union of menus \(B \cup C\) if and only if it is both preferred to \(B\) and to \(C\). For the proof, note that if one has \(A \succ (B \cup C)\) and, for instance, \(B \succeq A\), then it follows from \(Pr_4\) that \(B \succ (B \cup C)\); this together with the fact that \(B \subseteq (B \cup C)\) contradicts \(Pr_2\). For the converse, suppose we have \(A \succ (B \cup C)\) and, for instance, \(B \succeq A\); then by \(Pr_4\) we get \((B \cap C) \succ B\) and \(B \cup C \succ C\), that is, using \(Pr_3\), \(C \succ B\) and \(B \succ C\), contradicting \(Pr_1\).

Rule 2, which means that a union of menus \(B \cup C\) is preferred to a menu \(A\) iff one at least of \(B\) or \(C\) is itself preferred to \(A\), is a direct consequence of \(Pr_4\).

Rule 3 is immediate writing \(A = (A \cap B) \cup (A \cap B)\).

Similarly, the fourth rule is a direct consequence of \(\Omega = A \cup \overline{A}\), and the last one is an immediate application of \(Pr_2\).

### 2.1.2 The ranking function associated with a rational preference

Given a rational preference \(\succ\) on the \(S\), one easily checks that the relation \(\sim\) defined by: \(A \sim B\) iff \(A \succeq B\) and \(B \succeq A\) is an equivalence relation. The set of equivalence classes \([A]\) (\(A\) not empty) is totally ordered through the relation \(\succ\) defined by \([A] \succ [B]\) iff \(A \succ B\). The map that associates with every nonempty set its equivalence class provides a function \(\kappa\) from the set \(S - \{\emptyset\}\) onto a finite totally ordered set, and this set can be normalized to be the interval \([0, h - 1]\) defined \(\{0, 1, 2, ..., h - 1\}\). The function \(\kappa\) will be referred to as the utility function or the ranking function associated with the preference \(\succ\), and the integer \(\kappa(A)\) as the rank of \(A\). As for the integer \(h\), we will indifferently refer to it as the height of \(\kappa\), or the height of \(\succ\). We have \(\kappa(A) \succ \kappa(B)\) iff \(A \succ B\). Thus the menu \(A\) will be preferred to the menu \(B\) iff its utility is greater than that of \(B\). If \(a\) is an element of \(\Omega\), we shall write \(\kappa(a)\) for \(\kappa(\{a\})\).

**Lemma 1** For nonempty sets \(A\) and \(B\) one has

\[
\kappa(A \cup B) = \max(\kappa(A), \kappa(B))
\]
Proof: Suppose for instance that $\max(\kappa(A), \kappa(B)) = \kappa(A)$. We have then $A \succeq B$, hence, by $Pr_3$, $A \succeq (A \cup B)$. But it also follows from $Pr_1$ and $Pr_2$ that $(A \cup B) \succeq A$. This shows that $[A] = [A \cup B]$, so that these formulas have same rank.

The utility of a union is thus the greatest of the utility of its components. This important property in fact characterizes the class of utility functions that stem from rational preference orders: indeed, suppose given a function $\kappa$ from $\mathcal{S} - \{\emptyset\}$ onto $[0, h-1]$ that satisfies $\kappa(A \cup B) = \max(\kappa(A), \kappa(B))$. Define the relation $\succ$ on $\mathcal{S}$ by: $A \succ B$ iff $A = \emptyset$ or $\kappa(A) \succ \kappa(B)$. Then one easily checks that $\succ$ is a rational preference relation and that its associated ranking function is equal to $\kappa$.

This result together with the above lemma shows that there is a one to one mapping between the family of rational preference orders and that of the integral functions $\kappa$ that are defined on the power set of a finite sample set $\Omega$ and that satisfy the equality

$$\kappa(A \cup B) = \max(\kappa(A), \kappa(B))$$

Remark 1 Lemma 1 above shows that the ranking function associated with a rational preference is totally determined by its values on singletons.

Writing indeed $A$ as a union of singletons we have

$$\kappa(A) = \max_{a \in A} \kappa(a).$$

We have then, for any nonempty subsets $A$ and $B$ of $\Omega$, $A \succ B$ iff $\max_{a \in A} \kappa(a) > \max_{b \in B} \kappa(b)$, that is $A \succ B$ iff there exists an element $a$ of $A$ such that $\{a\} \succ \{b\}$ for all elements $b$ of $B$. This shows that the set of ordered pairs of elements of $\mathcal{S}$ is fully determined by the subset of ordered pairs of elements of $\Omega$, a result that can be directly retrieved from the first two derived rules. We will refer to this latter subset as to the elementary set of preferences of the preference relation $\succ$. If we identify elements $x$ of $\Omega$ with their corresponding singleton $\{x\}$, we see that the elementary set of preferences is just the restriction to $\Omega$ of the order relation $\succ$. We shall still denote by $\succ$ this restriction, writing $x \succ y$ for $\{x\} \succ \{y\}$. Note that $\succ$ is a strict partial modular order on $\Omega$: besides irreflexivity and transitivity, it satisfies
Theorem 1. Let $\succ$ be a strict partial modular order on $\Omega$. Still denote by $\succ$ the relation on the power set $\mathcal{S}$ of $\Omega$ defined by:

$A \succ B$ iff there exists an element $a$ of $A$ such that $a \succ b$ for all elements $b$ of $B$.

Then $\succ$ is a rational preference order that extends the original modular order $\succ$.

Proof: The extension of the original modular order may be interpreted as: menu $A$ is preferred to menu $B$ iff there exists an item in menu $A$ that is preferred to any item of $B$. The proof of the theorem is straightforward. Note that this result would still be valid if $\Omega$ were not supposed to be finite.

2.2 Rational preference orders and logical chains

An $\Omega$-chain $\Delta$ is a sequence of strictly embedded subsets of $\Omega$

$$D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_{h-1}.$$ 

In the sequel, we will always suppose that the last term of the chain is equal to $\Omega$. A chain $\Delta$ of the above form with $D_{h-1} = \Omega$ will be said of length $h$. Such a chain gives rise to a function $\kappa_\Delta$ from $\mathcal{S} - \{\emptyset\}$ to $[0, h-1]$ defined by:

$$\kappa_\Delta(A) = h - 1 - r_\Delta(A),$$

where $r_\Delta(A)$ is the first index $i$ such that $D_i$ intersects $A$. Thus the subsets of highest rank $h - 1$ are the ones that intersect $D_0$, while the subsets of rank 0 are those that have empty intersection with all the links of the chain but the last one. One has readily $\kappa_\Delta(A \cup B) = \max(\kappa_\Delta(A), \kappa_\Delta(B))$, so, by what precedes, $\kappa_\Delta$ is a ranking function. Its associated rational preference order $\succ_\Delta$ is defined by:

$$A \succ_\Delta B \text{ - that is if and only if } \kappa_\Delta(A) > \kappa_\Delta(B) \text{ iff } r(A) < r(B).$$
In other words, for nonempty sets, one has

\[ A \succ_{\Delta} B \text{ if and only if there exists a link } D_i \text{ that intersects } A \text{ and not } B. \]

We will refer to the order \(\succ_{\Delta}\) as to the order \textit{induced} or \textit{entailed} by the chain \(\Delta\). Note that the example of a rational order \(\succ_{\Delta}\) we gave in the preceding paragraph is just the order induced by a chain of length 2 with first link equal to \(D\). In the general case of a chain of length \(h\), the induced order may be interpreted as follows: elements of \(D_i\) are preferred to elements of \(D_{i+1}\) (that is, any element of \(D_i\) is preferred to any element of \(D_{i+1}\)). A nonempty subset set \(A\) of \(\Omega\) will be preferred to a subset \(B\) if its contains at least one element 'better' than any element of \(B\).

\textbf{Remark 2} Recalling that, for any element \(a\) of \(\Omega\), we write \(\kappa_{\Delta}(a)\) for \(\kappa_{\Delta}(\{a\})\), we see that \(D_i\) may be retrieved from the ranking function \(\kappa\) as the set of all elements \(a \in \Omega\) such that \(\kappa_{\Delta}(a) \geq h - 1 - i\).

The main interest in introducing \(\Omega\)-chains is that any rational preference order is induced by such a chain. More precisely we have the

\textbf{Theorem 2} Let \(\succ\) be a rational preference order on \(S\). Then there exists a unique chain \(\Delta\) such that \(\succ = \succ_{\Delta}\).

\textbf{Proof:} Let \(\kappa\) be the normalized ranking function associated with the preference \(\succ\) order, and \(h\) its height. For each integer \(i \in [0, h - 1]\) denote by \(D_i\) the set of all the elements of \(\Omega\) that have rank \(\geq h - 1 - i\). This yields a chain

\[ D = D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_{h-1} \]

of length \(h\), with last term equal to \(\Omega\). We have to prove that, for any nonempty subset \(A\) of \(\Omega\), \(\kappa_{\Delta}(A) = \kappa(A)\). Set \(j = r_{\Delta}(A)\). The set \(A\) then does not intersect \(D_{j+1}\) but intersects \(D_j\). This means that all elements of \(A\) have rank \(< h - j\) and that there exists an element of \(A\) with rank \(\geq h - j - 1\). We have therefore \(\kappa(A) = h - j - 1 = h - 1 - r_{\Delta}(A) = \kappa_{D}(A)\) as desired. The uniqueness of \(\Delta\) is an immediate consequence of the preceding remark.

The above theorem shows that in a finite environment the preferences, or the behavior, of a rational agent are always determined by a subjacent chain of subsets. If we restrict our attention to the elementary set of preferences of this agent, we see that an item \(a\) is preferred to an item \(b\) iff there exists an link \(D_i\) of \(\Delta\) such that \(a \in D_i\) and \(b \notin D_i\). This amounts to saying that \(a\) is preferred to \(b\) iff
1. For every index $k$, if $b \in D_k$ then $a \in D_k$

2. There exists an index $j$ such that $b \notin D_j$ and $a \in D_j$.

Under this form, it is possible to generalize this representation theorem in the non-rational case: this was done in the particular framework of finite propositional languages in [Freu98a], showing an analogue result for preference orders that do not satisfy the property of modularity expressed by $Pr_4$. In the set-theory framework we are interested in, this generalization is almost trivial:

**Theorem 3** Let $\succ$ be an strict partial order on a (not necessarily finite) set $\Omega$. Then there exists a subset $\Delta$ of the power set $\mathcal{S}$ of $\Omega$ such that, for all elements $a$ and $b$ of $\Omega$, one has $a \succ b$ iff

1. For every subset $D \in \Delta$, if $b \in D$ then $a \in D$

2. There exists a subset $X \in \Delta$ such that $b \notin X$ and $a \in X$

**Proof:** For any element $x$ of $\Omega$, let $D_x$ be the half-interval $D_x = \{ y \in \Omega$ such that $y = x$ or $y \succ x \}$. Let $\Delta$ be the set of all these $D_x$’s. We have to check that the preference $a \succ b$ holds iff the two above conditions are satisfied. Suppose first that we have $a \succ b$. By the transitivity of $\succ$, we see that if $b$ is a member of a set $D_x$, so must be $a$, so the first condition is satisfied. Moreover, by irreflexivity, we have clearly $a \in D_a$ and $b \notin D_a$, showing that the second condition also holds. Conversely, if we do not have $a \succ b$, either $a = b$, in which case the second condition is not satisfied, or we have $b \in D_b$ and $a \notin D_b$, contradicting therefore the first condition.

When $\Omega$ is a finite set, there is a one-to-one mapping between the strict partial orders on $\Omega$ and the ordering relations on $\mathcal{S}$ that satisfy conditions $Pr_0$ to $Pr_3$. The above theorem may be used to show that these relations are exactly those that are induced by arbitrary subsets $\Delta$ of $\mathcal{S}$.

Let us now illustrate theorem (2) by an example

**Example 1** Suppose the sample set consists on the two elementary items $p$ and $q$. We get all the rational preferences orders on $\mathcal{S}$ by considering the possible chains of subsets of $\Omega$. There are exactly two chains of length 2, $\Delta = \{ p \} \subset \{ p, q \}$ and $\Delta_1 = \{ q \} \subset \{ p, q \}$, and one chain of length 1, $\Delta_2 = \{ p, q \}$. These chains respectively induce on $\Omega$ the following modular orders
To write down the corresponding rational preference orders on $S - \{\emptyset\}$, we can apply the definition of the order induced by a chain or use the above array, remembering that menu $A$ is preferred to menu $B$ iff there exists an item in $A$ that is preferred to any item of $B$.

<table>
<thead>
<tr>
<th>$\succ$</th>
<th>$\succ_1$</th>
<th>$\succ_2$</th>
<th>rank</th>
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<td>$q$</td>
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<tr>
<td>$q$</td>
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<td>$q$</td>
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</table>

2.3 Revising rational preferences

2.3.1 How to handle the problem of preferences revision

Classically, a revision problem occurs when, disposing of a set of formulas (a knowledge base) that is supposed to represent all the information at disposal, it appears necessary to modify this set to take into account a new piece of information. A revision operation then consists in retracting from or adding to this basis one or several formulas. The famous A.G.M postulates [Alch85] have been the guiding line in A.I. for a number of searchers in an attempt to find optimal solutions. We have to point of, though, that this problem of classical revision does not fall within the scope of the present work: even if the reader finds some formal analogy with the classical formalism - e.g. similar terms and definitions - he should be aware that we are working now in a completely different perspective.

The problem of preference revising may be defined as the following one: we suppose the behavior of a given agent is represented by the set of his rational preferences and we decide modify this behavior; in this purpose, we want to withdraw, add, or replace one - or several - given preferences. Clearly, we always dispose of several ways to do so, and our problem is: on which grounds should we decide, which method, if any, should we adopt ? For instance, let $\Omega$ be the set of three samples $\{p, q, r\}$, and suppose we are given the following rational order in $S$:
We have thus \( \{p\} \succ \{r\} \). But suppose it appears afterwards that this relation should not be part of the agent’s preferences, or, even, that it should be replaced by the inverse one, \( \{r\} \succ \{p\} \). How should we change our model? Clearly, as this corresponds to an elementary induced order in which we should have \( r \succ p \), we have exactly five solutions, corresponding to the extension of the five following elementary modular orders

\[
\begin{array}{cccccc}
\succ_1 & \succ_2 & \succ_3 & \succ_4 & \succ_5 & \text{rank} \\
\hline
r & r & q & r & r & 2 \\
p & q & r & p & p & 1 \\
p & q & q & p & p & 0 \\
\end{array}
\]

Now, which one are we going to choose, and on what grounds shall we determine our choice? Note that any of these rankings induces the desired preference but does it at some expense, inducing changes that do not seem to be necessary: for instance, in the three first one, we get \( \{r\} \succ \{q\} \), which we did not have previously, while we only wanted to add \( \{r\} \succ \{p\} \), and in the two last ones, we get \( \{q\} \succ \{p\} \), reversing, without apparent necessity the original \( \{p\} \succ \{q\} \). Thus it is clear that none of the above modifications may be considered as the (unique) right solution as they all imply some unnecessary loss of the original information. We see that even in the simplest situation of a sample set with only three items the preference revision problem does not offer quite an obvious solution.

Things work differently if one addresses this problem in the perspective of subset chains: indeed, by what we saw in the preceding section, any rational order may be represented by a well-determined chain. So the operation of transforming this preference into another one boils down to a chain-transformation problem: given a subset chain \( \Delta \) that induces the particular preference \( A \succ B \), transform it in a reasonable fashion into a chain \( \Delta' \) that will no more induce \( A \succ B \), (or that will induce \( A' \succ B' \)). Of course, we have to be a bit more precise concerning this ‘reasonable’ fashion and examine more closely what properties we should expect from an ideal transformation. For this it will be useful to make a distinction between two
different problems, that of the *contraction* of a rational preference set by a preference \( A \succ B \) and that of its *revision* by this preference:

a) The **contraction** problem occurs when one wants to withdraw a particular preference \( A \succ B \) from the given set of rational preferences. This problem amounts, given, a chain \( \Delta \) that entails \( A \succ B \), to building a new chain \( D \div (A \succ B) \) that does not anymore entail this particular preference.

b) We shall talk of the **revision** of a set of rational preferences by a preference \( A \succ B \) when we deal with the problem of adding this particular preference to the given set. In other words, starting from a chain \( \Delta \) that does not entail \( A \succ B \), we are looking for a chain \( D\ast(A \succ B) \) that entails this preference.

Considering both operations, there exists a few natural and elementary principles that we should observe. Naturally, we first expect **success**, in the sense that the contraction by \( A \succ B \) should not anymore entail this preference, while revision by \( A \succ B \) should induce this preference. But apart from this quite legitimate request, there is another principle we would like to conform with, that of **minimal change**, which requires to change as little as possible from the behavior of the agent, in as much as this behavior is represented by his set of preferences. More precisely we shall work with the following constraints:

1) If \( \Delta \) does not entail \( A \succ B \), then \( D \div (A \succ B) = D \).
2) If \( \Delta \) entails \( A \succ B \), then \( D\ast(A \succ B) = D \).
3) Contracting by \( A \succ B \) does not add any new preferences.
4) Contracting by \( A \succ B \) eliminates only the preferences that necessarily imply \( A \succ B \).
5) Revising by \( A \succ B \) eliminates only the preferences that are incompatible with \( A \succ B \).
6) Revising by \( A \succ B \) doesn’t unnecessarily adds new preferences.

The two first constraints translate the principles of **minimal change** and **success** and are self-explanatory. The third one states that if one desires to withdraw a particular preference from the set of preferences of an agent, this should be done without adding any new preference. Indeed, the introduction of a new preference is supported by no justification and would be thus quite arbitrary. The fourth rule recalls that the aim of the contraction is to with-
draw $A \succ B$ and nothing else, when possible. If this is not possible, then only a minimal number of preferences have to be withdrawn. Similarly the two last constraints define a principle of minimal change for revision: it may well be the case that a new preference cannot be added alone in the agent's preference set, and that it implies moreover to withdraw some of the initial preferences. But in any cases, the operation of revision should be performed in such a way that no unnecessary changes will be operated.

This principles being stated, it will be easy to apply them in both cases of contraction and revision, making simply use of the following

**Lemma 2** Let $\Delta$ and $\Delta'$ be two $\Omega$-chains, with induced preference orders $\succ_{\Delta}$ and $\succ_{\Delta'}$. Then one has $\succ_{\Delta} \subseteq \succ_{\Delta'}$ iff $\Delta$ is a sub-chain of $\Delta'$, that is iff every link of $\Delta$ is a link of $\Delta'$.

**Proof:** Suppose first that $\Delta$ is a sub-chain of $\Delta'$. If we have $A \succ_{\Delta} B$ for two menus $A$ and $B$, this implies by definition that there exists a link of $\Delta$ that intersects $A$ and does not intersect $B$. Since this link is also a member of the chain $\Delta'$, we have readily $A \succ_{\Delta'} B$.

Conversely, let us show that $\succ_{\Delta} \subseteq \succ_{\Delta'}$ implies that $\Delta$ is a sub-chain of $\Delta'$. Recall that, as shown in the proof of theorem (2), the i-th link $D_i$ of the chain $\Delta$ is the set of all elementary items $x \in \Omega$ such that $\kappa_{\Delta}(x) \geq h - 1 - i$, where $h$ is the length of $\Delta$. Let $z$ be an elementary item with minimal $\kappa_{\Delta}$-rank among the samples that have $\kappa_{\Delta}$ rank equal to $h - 1 - i$. The link $D_i$ is then the set of all items $x$ such that $\kappa_{\Delta}(x) \geq \kappa_{\Delta}(z)$. Set $j = h' - 1 - \kappa_{\Delta'}(z)$. We claim that $D_i$ is precisely equal to $D'_j$, and is thus the j-th link of $\Delta'$. By the choice of $z$, and the fact that $\succ_{\Delta} \subseteq \succ_{\Delta'}$, we have indeed $h' - 1 - j = \kappa_{\Delta'}(z) \leq \kappa_{\Delta'}(x)$ for any sample $x$ with $\kappa_{\Delta}$ rank $\geq h - 1 - i$. Conversely, if $x$ is a sample such that $\kappa_{\Delta}(x) \geq h' - 1 - j$, we have $\kappa_{\Delta}(z) \leq \kappa_{\Delta}(x)$, and it follows that $z \preceq_{\Delta'} x$ and therefore, by our hypothesis, that $z \preceq_{\Delta} x$. This means that $h - 1 - i \leq \kappa_{\Delta}(x)$. We have therefore proven that, given an elementary item $x$, one has $\kappa_{\Delta}(x) \geq h - 1 - i$ iff $\kappa_{\Delta}(x) \geq h' - 1 - \kappa_{\Delta'}(z)$. This shows that $D_i = D'_j$, and the proof of the lemma is complete.

### 2.3.2 Chain contraction

We suppose now that the preferences of an agent are given by a chain

$$\Delta = D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_{h-1}$$
with last link equal to $\Omega$. We want to build a new chain $\Delta' = D \div (A \succ B)$ that does not induce $A \succ \Delta' B$. Clearly, for this to be possible, we have to suppose that $B \neq \emptyset$. Taking into account the principles exposed in the precedent paragraph, it follows from Lemma (2) that the chain $\Delta'$ we are looking for should be a maximal sub-chain of $\Delta$ that does not entail $A \succ B$.

Denote by $i = r(A)$ the first index such that $D_i$ intersects $A$, and by $j = r(B)$ the first index such that $D_j$ intersects $B$. Recall that a chain $\Gamma$ entails the preference $A \succ B$ iff $r(A) < r(B)$. If $j \leq i$, $\Delta$ does not entail the preference $A \succ B$, we just set $\Delta' = \Delta$. If $i < j$, any sub-chain of $\Delta$ that contains a link $D_s$ with $i \leq s < j$ will still induce $A \succ B$. We have therefore to remove of the chain $\Delta$ all its links $D_s$ such that $r(A) \leq s < r(B)$. It follows that the (unique) solution to the contraction problem is the chain:

$$D_0 \subset D_1 \subset \ldots \subset D_{i-1} \subset D_i \subset \ldots \subset D_{h-1} = \Omega,$$

where $i$ is the first index such that $D_i \cap A \neq \emptyset$ and $j$ is the first index such that $D_j \cap B \neq \emptyset$.

Let us compute the new rank $\kappa' = \kappa_{\Delta \div (A \succ B)}$ induced by the contracted chain $\Delta' = \Delta \div (A \succ B)$. We have, for any subset $X$, $\kappa(X) = h - 1 - r(X)$ and $\kappa'(X) = h' - 1 - r'(X)$. Note first that we have $h' = h - (j - i)$, that is $h' = h - (\kappa(A) - \kappa(B))$. We get herefore $r'(X) = r(X)$ if $r(X) \leq r(A)$, that is if $\kappa(A) \leq \kappa(X)$, $r'(X) = r(A)$ if $r(A) < r(X) \leq r(B)$, that is if $\kappa(B) \leq \kappa(X) < \kappa(A)$, and $r'(X) = r(X) - (\kappa(A) - \kappa(B))$ if $\kappa(X) < \kappa(B)$.

It follows that the new ranking $\kappa'$ is given by

$$\kappa'(X) = \begin{cases} 
\kappa(X) - [\kappa(A) - \kappa(B)] & \text{if } \kappa(A) \leq \kappa(X) \\
\kappa(B) & \text{if } \kappa(B) \leq \kappa(X) < \kappa(A) \\
\kappa(X) & \text{if } \kappa(X) < \kappa(B)
\end{cases}$$

Clearly the contraction operation thus defined fully meets the constraints of success and of minimal change that were required for any 'reasonable' contraction operation.

**Example 2** The preferences of the agent are now given by the rational order $\succ$. 

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Suppose we want to contract $\succ$ by $\{p\} \succ \{r\}$. We first have to determine the underlying chain $\Delta$. The original associated elementary modular order is given by

\[
\begin{array}{ccc}
\{p\} & \{p,q\} & \{p,r\} & \{p,q,r\} \\
\{q\} & \{q,r\} \\
\{r\} \\
\emptyset \\
\end{array}
\]

and the chain $\Delta$ is just $\{p\} \subset \{p,q\} \subset \{p,q,r\}$. With the preceding notations, we have have $i = 0$ and $j = 2$, and the contracted chain therefore boils down to the trivial chain $\{p,q,r\}$. The modular order on $\Omega$ is the trivial one, where all three items have rank 0, and the associated rational order is just: $A \succ B$ iff $B = \emptyset$. Note that two other choices were possible in order to give equal rank to $p$ and $q$: in the first one, we could give rank 0 at $p$ and $r$ and put $q$ at rank 1, whilst, in the second one, $p$ and $r$ could get rank 1 and $q$ rank 0. The chain contraction we chose disregards these solutions as not economical: indeed, in the first one, the original preference $p \succ q$ would be unnecessarily reversed, and, in the second one, it is the preference $q \succ r$ that would be reversed.

Remark 3 It is possible to perform a contraction by simply working on the elementary set of preferences on the sample set $\Omega$. Suppose indeed we have to contract by $A \succ B$. Let $i = r(A)$ and $j = r(B)$. There exist then elements $a \in A$ and $b \in B$ such that $r(a) = i$ and $r(b) = j$ and we have $a \succ b$. Recall that the $k$-th link $D_k$ of the chain $\Delta$ that诱导 $\succ$ is the set of all elementary samples $x$ such that $r(x) \leq k$. After the contraction has been performed, we see that we do not have anymore $a \succ b$ in the resulting elementary preference set, exactly as if we had performed in this set a contraction by $a \succ b$. It is straightforward to check that it is equivalent to contract a rational preference order by a preference $A \succ B$ or to contract the corresponding elementary modular order by $a \succ b$, where $a$ and $b$ are elements of $A$ and $B$ with maximal rank.
2.3.3 Chain revision

Our problem is now to add a given preference, say \( A \succ B \), to the set of rational preferences that reflects the behavior of an agent. Clearly, the procedure will differ, depending on whether or not this preference stands in contradiction with the original set of preferences. The intuitive notion of contradiction can be made more precise: naturally, such a contradiction would result of the presence of \( B \succ A \) among the given set of preferences of the agent, and there would also be a contradiction if the original set contained the preference \( A \succ \mathcal{B} \), since, by the first derived rule, \( A \succ B \) and \( A \succ \mathcal{B} \) lead to \( A \succ \Omega \), which contradicts the fifth derived rule. We would still have a contradiction, if the set of preferences contained the preference \( B \succ A \cap \mathcal{B} \), for this latter inequality together with \( A \succ B \) implies \( (A \cup B) \succ A \cup B \). As a matter of facts, it will turn out that only this latter case only poses a problem. We shall therefore first examine the situation where the preference \( B \succ A \cap \mathcal{B} \) is not induced by the chain

\[
D = D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_{h-1}
\]

that is supposed to represent the agent’s behavior. Note that this requires that \( A \) is not a subset of \( B \).

In this simple case, the usual terminology is that of an expansion of \( \Delta \) by \( A \succ B \). Similarly to the notation used in classical revision theory, we shall denote by \( D + (A \succ B) \) the result of this chain expansion. As follows from the principle of minimal change and Lemma (2), \( D + (A \succ B) \) should be the smallest super-chain of \( \Delta \) that induces \( A \succ B \), if such a chain exists.

Let \( i = r(\alpha) \). Since we supposed that \( \Delta \) does not induce the preference \( B \succ A \cap \mathcal{B} \), we do not have either \( B \succ A \), as results from the last derived rule. It follows that \( D_{i-1} \subseteq \mathcal{B} \). Observe furthermore that \( D_i \) intersects the set \( A \cap \mathcal{B} \): otherwise we would have \( A \succ (A \cap \mathcal{B}) \), hence \( B \succ (A \cap \mathcal{B}) \) as follows from the third and fifth derived rules.

Consider now the following chain \( \Delta' \)

\[
D_0 \subset D_1 \subset \ldots \subset D_{i-1} \subset D_i \cap \mathcal{B} \subset D_i \subset \ldots \subset D_{h-1} = \Omega.
\]

obtained from \( \Delta \) by simply adding the link \( D_i \cap \mathcal{B} \). We claim that this chain fulfils all the requirements that were expected from the expansion of \( \Delta \) by \( A \succ B \):

- \( \Delta' \) is indeed a chain since \( D_{i-1} \subseteq D_i \cap \mathcal{B} \)
• $\Delta'$ induces $A \succ B$ because $D_i \cap \overline{B}$ intersects $A$ and does not intersect $B$

• $\Delta'$ is a minimal extension of $\Delta$ in the sense that
  1. One has readily $\Delta' = \Delta$ if $\Delta$ primitively entailed $A \succ B$
  2. Only one link was added to the original chain
  3. The link that was added has minimal strength: one checks easily indeed that if adding a link $D$ to $\Delta$ entails the preference $A \succ B$, then it must be the case that $D \subseteq D_i \cap \overline{B}$

We shall refer to this chain as 'the' expansion of $\Delta$ by $A \succ B$ and denote it by $D + (A \succ B)$.

Let us compute the new rank $\kappa'$ induced by this chain. We have $h' = h + 1$, and, for any menu $C$:

- $r'(C) = r(C) + 1$ if $r(C) < i$
- $r'(C) = r(C)$ if $r(C) = i$
- $r'(C) = i + 1$ if $r(C) = i$ and $C$ does not intersect $D_i \cap \overline{B}$

It follows that

$$
\kappa'(C) = \begin{cases} 
\kappa(C) + 1 & \text{if } \kappa(C) > \kappa(\alpha) \\
\kappa(C) & \text{if } \kappa(\alpha) > \kappa(C) \\
\kappa(A) + 1 & \text{if } \kappa(C) = \kappa(A) \text{ and } \kappa(C \cap \overline{B}) \geq \kappa(A) \\
\kappa(A) & \text{if } \kappa(C) = \kappa(A) \text{ and } \kappa(C \cap \overline{B}) < \kappa(A) 
\end{cases}
$$

The row of the elementary items that had rank equal to $\kappa(A)$ has just split into two rows: the samples that are elements of $B$ form a new rank, one notch downwards. The other ones don’t move.

**Example 3** We take again $\Omega = \{p, q, r\}$, and suppose we are given the following rational order in $S$:

$${p} \quad \{p, q\} \quad \{q\} \quad \{q, r\} \quad \{p, r\} \quad \{p, q, r\} \quad 1 \\
\{r\} \quad \{p, r\} \quad 0 \\
\emptyset \quad \emptyset$$

This preference order is induced by the chain $\{p, q\} \subset \{p, q, r\}$. Suppose we want to revise it by $\{p, q, r\} \succ \{p, r\}$. Note that $\{p, q, r\} \cap \{p, r\} = \{q\}$. As
we do not have \( \{p, r\} \succ \{q\} \), the expansion by \( \{p, q, r\} \succ \{p, r\} \) is possible. The expanded chain is \( \{p, q\} \cap \{q\} \subset \{p, q\} \subset \{p, q, r\} \), that is

\[
\{q\} \subset \{p, q\} \subset \{p, q, r\}.
\]

The revised preference order is therefore

\[
\{p, q\} \quad \{q\} \quad \{q, r\} \quad \{p, q, r\} \\
\{p\} \quad \{p, r\} \\
\{r\} \quad 0 \\
\emptyset \quad -
\]

We have \( \{p, q, r\} \succ \{p, r\} \) as desired. Note that in the original elementary modular order on \( \Omega \), \( r \) had rank 0 and \( p \) and \( q \) had rank 1. The new ranking put \( q \) at rank 2, leaving the other elements unchanged.

**Remark 4** The condition that one does not have \( B \succ A \cap \overline{B} \) is equivalent to the condition: for every element \( b \in B \) with maximal rank there exists an element \( a \in A \) with maximal rank, \( a \notin B \), such that \( a \succeq B \). When this is the case, the expansion by \( A \succ B \) may be directly performed through a simple expansion by \( a \succ b \) on the elementary preference set. For instance, in the above example, the elementary preference order on \( \Omega \) is given by \( \kappa(p) = \kappa(q) = 1 \), and \( \kappa(r) = 0 \). To perform the expansion by \( \{p, q, r\} \succ \{p, r\} \), we only have to expand the original chain \( \{p, q\} \subset \{p, q, r\} \) by \( q \succ p \). This leads to

\[
\{p, q\} \cap \{p\} \subset \{p, q\} \subset \{p, q, r\},
\]

that is to

\[
\{q\} \subset \{p, q\} \subset \{p, q, r\}.
\]

We finally turn to the more complicated case where the given chain \( \Delta \) induces the preference \( B \succ A \cap \overline{B} \). As we noticed, it is not anymore possible to just make an expansion of \( \Delta \) by \( A \succ B \). The natural way to add this latter preference to our agent’s scheme is thus to first retract the preference \( B \succ A \cap \overline{B} \) through our contraction procedure, and then to make an expansion by \( A \succ B \). As follows from the fifth derived rule, a necessary condition for this to be possible is that \( A \) is not a subset of \( B \). We shall see that this is also a sufficient condition. Using the notations of classical
revision theory, we shall denote by $\Delta \star (A \succ B)$ the resulting chain, that is $\Delta \star (A \succ B) = (\Delta \div (B \succ A \cap \overline{B})) + (A \succ B)$.

Set $i = r(B)$ and $j = r(B \succ A \cap \overline{B})$. The contraction by $B \succ A \cap \overline{B}$ gives rise to the chain

$$D_0 \subset \ldots \subset D_{i-1} \subset D_j \subset \ldots \subset \Omega.$$ 

The first link of this chain that intersects $A$ is now $D_j$. Expanding by $A \succ B$ therefore provides the revised chain $D \star (A \succ B)$ that is equal to

$$D_0 \ldots \subset D_{i-1} \subset D_j \cap \overline{B} \subset D_j \subset D_{j+1} \ldots \subset \Omega.$$ 

The new rank $\kappa' = \kappa \star (A \succ B)$ is given, for any elementary item $z$ by:

$$\kappa'(z) = \begin{cases} 
\kappa(z) - \kappa(B) - \kappa(A \cap \overline{B}) & \text{if } \kappa(z) > \kappa(B) \\
\kappa(z) & \text{if } \kappa(z) < \kappa(A \cap \overline{B}) \\
\kappa(A \cap \overline{B}) & \text{if } \kappa(A \cap \overline{B}) \leq \kappa(z) \leq \kappa(B) \text{ and } z \notin B \\
\kappa(A \cap \overline{B}) - 1 & \text{if } \kappa(A \cap \overline{B}) \leq \kappa(z) \leq \kappa(B) \text{ and } z \in B 
\end{cases}$$

Example 4 Let us take back the example of page 12

$$\{p\} \{p, q\} \{p, r\} \{p, q, r\} \ 1$$ $$\{q\} \{q, r\} \{r\} \ 0$$ $$\emptyset \ -$$

Suppose we want to revise by $\{r\} \succ \{p\}$. The underlying chain is

$$\{p\} \subset \{p, q, r\}$$

Using the above notations, we get $A \cap \overline{B} = \{r\}$, $i = 0$ and $j = 1$. It follows that the revised chain is

$$\{q, r\} \subset \{p, q, r\}.$$ 

The revised elementary modular order is therefore the order $\succ_{4}$ of page 12.

2.4 Iterated revision

Up till now, we have dealt with the problem of revising a set of preferences $P$ in order to incorporate in it a single preference $A \succ B$. The problem of iterated revision comes when, this having been done, it appears desirable or necessary to revise the new preference set by a second preference, say $C \succ D$. 

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Theoretically, one should process naturally by revising, through the method exposed above, the set $P \star (A \succ B)$ by $C \succ D$. Thus, the solution would be simply to take as resulting preference set the set $(P \star (A \succ B)) \star (C \succ D)$. If then the preference $A \succ B$ happens to disappear in the process, this is totally justified by the choice of the sequence in which we decided to make the revisions. Revising by $C \succ D$ only after the revision by $A \succ B$ has been performed is interpreted as: the requirement to get the preference $C \succ D$ follows that concerning the preference $A \succ B$, and must therefore be considered as more important or more reliable than this latter. In this perspective, where former information or constraint may become obsolete, there is no harm and no contradiction in giving up a preference that contradicts the new one, even if the former was inserted in the initial preference set by means of a revision process.

Nevertheless, it may well be the case that one wishes to add both preferences $A \succ B$ and $C \succ D$ to the initial preference set. We may think of simply performing a revision by $A \cap C \succ B \cup D$, as this preference entails both preferences $A \succ B$ and $C \succ D$, but this method may be considered as an excessive one - think for instance to the case where the set $A \cap C$ is empty.

To fix a bit the ideas and show the type of problems we have to deal with, it may be useful to consider some elementary examples:

### 2.4.1 Fruit puzzles

**Puzzle 1** We dispose of a sample set $\Omega$ with four elements, an apple $a$, a banana $b$, a cherry $c$ and a date $d$. We would like to complete the preferences of a rational agent who is known to prefer dates to bananas and bananas to both apples and cherries. In other words, we want to build a chain that will induce an order $\succ$ such that one has $d \succ b$ and $b \succ \{a, c\}$. One way to construct such a chain would be to start with the trivial chain $\Omega$ and successively revise it by $d \succ b$ and $b \succ \{a, c\}$. Now, after the first revision has been performed (here an elementary expansion), we obtain the chain $\{a, c, d\} \subset \Omega$. To revise it with the second preference $b \succ \{a, c\}$, we cannot proceed by a simple expansion since we have $\{a, c\} \succ (\{b\} \cap \{a, c\})$: we have first to get rid of this preference and only then can we expand by $b \succ \{a, c\}$. Using the results of the preceding section, we finally get as resulting chain $\{b, d\} \subset \Omega$... and realize we have lost the first of our preferences, $d \succ b$. 


In fact, it is nevertheless possible to get this latter preference back and integrate both our desired preferences into a rational preference set. Observe indeed that, starting from our latter chain \( \{b, d\} \subset \Omega \), all we have to do is to proceed to a third revision. As may be immediately seen, a simple expansion by \( d \succ b \) does the job and yields the chain \( \{d\} \subset \{b, d\} \subset \Omega \) that induces both desired preferences. Note that this chain may be also obtained by two revisions instead of three if we first begin by \( b \succ \{a, c\} \) and only then revise by \( d \succ b \).

**Puzzle 2** Still working with the set \( \Omega = \{a, b, c, d\} \), we want now to build a rational preference order such that both preferences \( \{a, b\} \succ \{c\} \) and \( \{c, d\} \succ \{a, b, d\} \) hold. The expansion of the trivial chain by the first preference yields the chain

\[
\{a, b, d\} \subset \Omega.
\]

The subsequent revision by the second preference yields the chain

\[
\{c\} \subset \Omega.
\]

Again, the first preference has been lost in the process. We may try, like in our preceding puzzle, to perform the revision of this latter chain by \( \{a, b\} \succ \{c\} \), but this leads back to the chain

\[
\{a, b, d\} \subset \Omega.
\]

Clearly, it is impossible to get both desired preferences through iterated revision. In fact it is impossible to find a preference set that would contain both preferences \( \{a, b\} \succ \{c\} \) and \( \{c, d\} \succ \{a, b, d\} \): indeed, it follows from the second one that the element of highest rank in \( \{c, d\} \) is \( c \) and has greater rank than \( a, b \) and \( d \), which contradicts the first preference.

**Puzzle 3** We start now from the non-trivial following chain \( \Delta \)

\[
\{a, b, c\} \subset \Omega
\]

that represents the preferences of an agent who would prefer any of the three first fruits to a date. Suppose we want to modify this chain so that both preferences \( \Omega \succ \{b, c\} \) and \( \Omega \succ \{a, c\} \) hold. The expansion by the first preference leads to the chain \( \Delta_1 \):

\[
\{a\} \subset \{a, b, c\} \subset \Omega.
\]
The revision by $\Omega \succ \{a, c\}$ first requires a contraction by $\{a, c\} \succ \{b, d\}$, leading back to $\Delta_1$, then an expansion, which yields the chain $\Delta_2$

$$\{b\} \subset \{a, b, c\} \subset \Omega.$$ 

The first preference $\Omega \succ \{b, c\}$ is no more entailed by this chain. We may try a revision of $\Delta_2$ by this preference, but then we get again the chain $\Delta_1$, losing this time our second preference. Hence it is clear that we cannot get both preferences from the original chain $\Delta$ applying the revision processes which were described in the preceding paragraphs. Note though that these two preferences may well coexist: they are for instance induced by the chain $\Delta'$:

$$\{d\} \subset \{a, b, c\} \subset \Omega.$$ 

This example shows that the revision process that we defined in the preceding paragraphs may not give a satisfactory answer to the problem of both iterating a revision and preserving it at the same time. The question we shall address now is: given a rational preference set and four menus $A, B, A'$ and $B'$, under what conditions can we be sure to obtain, through our chain revision procedure, both preferences $A \succ B$ and $A' \succ B'$?

### 2.4.2 Possible and impossible iterated revision

The preference set of an agent being given by the chain

$$\Delta = D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_{h-1}$$

with last link equal to $\Omega$, our purpose is now to determine whether it is possible to transform this chain by a succession of iterated revisions so that the resulting chain entails both preferences $A \succ B$ and $A' \succ B'$.

We may restrict ourselves to the case where $\Delta$ already entails $A \succ B$ because, anyway, this will be the case for the revised chain $\Delta' = \Delta \ast (\Delta \succ B)$.

We set $i = r(A) = r(A \cap \overline{B})$, $j = r(B)$, $j' = r(B')$ and $k = r(A' \cap \overline{B'})$. We have $i > j$. We shall adopt the ‘impossibility hypothesis’ that no iteration of the procedure described in the preceding section is sufficient to turn $\Delta$ into a chain that would induce at the same time the two preferences $A \succ B$ or $A' \succ B'$.

The original chain $\Delta$ then necessarily entails $B' \succ (A' \cap \overline{B})$, otherwise revision by $A' \succ B'$ would be a simple expansion, preserving $A \succ B$. We
have therefore \( j' < k \) and the revised chain \( \Delta \star (A' \succ B') \) is of the form

\[
D_0 \subset \ldots \subset D_{j'-1} \subset D_k \cap \overline{B} \subset D_k \subset \ldots D_{h-1}.
\]

By our hypothesis, this chain does not anymore entail \( A \succ B \), and this implies readily that \( j' \leq i < j \leq k \). Moreover, this chain entails\( B \succ (A \cap \overline{B}) \), since otherwise, making an expansion by \( A \succ B \) would yield both preferences. It follows that the link \( D_k \), which intersects \( B \), cannot be the first one that meets \( B \), since it also intersects \( A \cap \overline{B} \). Since \( D_{j'-1} \) does not intersect \( B \), we must have \( (D_k \cap \overline{B}) \cap B \neq \emptyset \), and \( D_k \) is thus the first link of the revised chain that intersects \( B \). Also, as we saw, this link does not intersect \( A \cap \overline{B} \).

Revising then by \( A \succ B \) yields the chain \( (\Delta \star (A' \cap \overline{B})) \star (A \succ B) \) of the form

\[
D_0 \subset \ldots \subset D_{j'-1} \subset D_k \cap \overline{B} \subset D_k \subset \ldots D_{h-1}.
\]

Applying again our hypothesis concerning the impossibility to get at the same time our two preferences, we see that this chain does not entail \( A' \succ B' \) but entails \( B' \succ (A' \cap \overline{B}) \). Thus \( D_k \cap \overline{B} \) does not intersect \( A' \cap \overline{B} \). As a last attempt to revise this chain by \( A' \cap \overline{B} \) leads back to the previous \( \Delta \star (A' \cap \overline{B}) \), we see that our impossibility condition is equivalent to the conjunction the following conditions: \( j' \leq i < j \leq k \), \( D_k \) intersects \( B \cap \overline{B} \), and \( D_k \) has an empty intersection with \( A \cap \overline{B} \cap \overline{B} \) as well as with \( A' \cap \overline{B} \cap \overline{B} \). Translating all this in terms of preference ranking, we obtain the

**Theorem 4** Let \( P \) be a complete set of preferences, \( A \succ B \) a preference of \( P \) and \( A' \succ B' \) an arbitrary preference. Then it is possible to modify \( P \) through iterated revisions and obtain both preferences \( A \succ B \) and \( A' \succ B' \) iff one at least of the following conditions is satisfied:

1. \( A \succ B' \)
2. \( A' \cap \overline{B} \succ B \)
3. \( A' \cap \overline{B} \preceq (A \cup A') \cap \overline{B} \cap \overline{B} \).

The above result therefore provides a criterium to test whether two given preferences may be together incorporated through iterated revisions in the preference set of an agent. If none of the three conditions stated in theorem 4 is satisfied, it is impossible to revise the original preference set and get both preferences together. It may be the case that these two new preferences are
mutually exclusive, as happened in puzzle 2, but, as we saw in puzzle 3, it may also be the case that the revision procedure we used is inadequate to treat the general problem of multi-revision. In its all generality, this problem amounts to the following: given the preference set $\mathcal{P}$ of an agent together with a consistent set $K$ of preferences, determine a procedure to build a new preference set $\mathcal{P}'$ that would include $K$ and differ as little as possible from $\mathcal{P}$. For instance, we may take the example where $K$ consists of two preferences $A \succ B$ and $A' \succ B'$, which was treated by the above theorem. We can also adopt a slightly different point of view, considering that only one preference has to be added at each time to $\mathcal{P}$ (here for example $A' \succ B'$), but that we have the constraint to preserve some subset of $\mathcal{P}$ (e.g. $A \succ B$).

It is doubtful that a generalization of theorem 4 may provide practical results in the case where $K$ has more than two or three elements. But, above all, it seems that this constraint problem (preserving a given preference base $K$) does not fall in the context of iterated revision, where the new preferences to be added or retracted are taken one by one in a sequence that is the essential part of the revision process, the last preference to be treated being supposed to be more reliable than the preceding ones. Thus the constraint problem requires a study of its own for which the results established in the preceding sections are of little help.
Part II
The semantics of choices

3 The representation of choice functions by subsets chains

3.1 Arrow conditions on choice functions and rational preferences

A choice function on $$\Omega$$ is a function $$f : \wp(\Omega) \to \wp(\Omega)$$ that chooses, for any set $$X$$, a set of best or preferred elements of $$X$$. These functions are required to satisfy a certain number properties. For a review and a justification of these properties, see for instance [Leh01]. In this paper we shall consider only choice functions that, apart from the two basic properties of inclusion,

$$\forall X \subseteq \Omega, \ f(X) \subseteq X,$$

and consistency,

$$f(X) \neq \emptyset \text{ whenever } X \neq \emptyset$$

satisfy the Arrow condition:

$$\text{if } X \subseteq Y \text{ and } X \cap f(Y) \neq \emptyset, \text{ then } X \cap f(Y) = f(X)$$

Choice functions that satisfy these three conditions will be referred to as rational choice functions. When $$\Omega$$ is the set of worlds attached to a finite propositional language $$\mathcal{L}$$, there exists a tight link between rational choice functions, plausibility measures on $$\wp(\Omega)$$ and rational inferences on $$\mathcal{L}$$: the reader may consult [Leh.01], [Rott.93] or [Rott.01] for an exhaustive study.

We will be interested in the links between rational choice functions and rational preferences. Our first result is the

**Theorem 5** Let $$f$$ be a rational choice function. Define the relation $$\succ$$ on the language $$S = \wp(\Omega)$$ by: $$A \succ B$$ iff $$B \cap f(A \cup B) = \emptyset$$. Then $$\succ$$ is a rational preference order.

In the light of this theorem, the subset $$A$$ is preferred to $$B$$ iff the list of the best elements of $$A \cup B$$ contains no element of $$B$$. 

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Proof: (we slightly adapt a proof given in [Leh.01]) As usual, we write
\( A \preceq B \) if it is not the case that \( A \succ B \). We have to check the following:

\textbf{Pr}_0 \ A \succ \emptyset \ for \ any \ set \ A. \ Obvious.

\textbf{Pr}_1 \ if \ A \succ B \ then \ B \preceq A \ for \ non-empty \ sets \ A \ and \ B. \ Suppose \ that
A \succ B \ and \ B \succ A. \ Then \ (A \cup B) \cap f(A \cup B) = \emptyset, \ contradicting
inclusion \ + \ consistency.

\textbf{Pr}_2 \ if \ A \succ C \ and \ A \subseteq B \ then \ B \succ C: \ Suppose \ that \ we \ do \ not \ have
B \succ C, \ so \ that \ C \cap f(B \cup C) \neq \emptyset. \ Applying \ the \ Arrow \ condition
to \ the \ set \ A \cup C, \ considered \ as \ a \ subset \ of \ B \cup C, \ yields \ f(A \cup C) =
(A \cup C) \cap f(B \cup C), \ contradicting \ the \ fact \ that \ C \cap f(A \cup C) = \emptyset.

\textbf{Pr}_3 \ if \ A \cup B \succ B \ then \ A \succ B: \ immediate.

\textbf{Pr}_4 \ if \ B \preceq A \ and \ C \succ A \ then \ C \succ B: \ We \ have \ A \cap (f(A \cup C) =
\emptyset. \ Let \ us \ first \ show \ that, \ together \ with \ the \ hypothesis \ B \preceq A, \ this
implies \ (A \cup B) \cap f(A \cup B \cup C) = \emptyset. \ Indeed, \ if \ this \ were \ not \ the \ case, \ we \ would \ have \ by \ Arrow \ condition \ f(A \cup B) = f(A \cup B \cup C). \ Note \ that, \ by \ Arrow \ condition \ again, \ we \ always \ have \ f(A \cup B \cup C) \subseteq f(A \cup C) \cup (B - A), \ and \ the \ last \ equality \ would \ therefore \ lead \ to \ A \cap f(A \cup B) \subseteq A \cap (f(A \cup C) \cup (B - A)) = \emptyset, \ hence \ to \ B \succ A, \ a
contradiction. \ We \ see \ therefore \ that \ we \ have \ (A \cup B) \cap f(A \cup B \cup C) = \emptyset.
It \ follows \ that \ f(A \cup B \cup C) \subseteq C \subseteq C \cup B. \ Applying \ Arrow \ condition
to \ C \cup B \subseteq C \cup A \cup B, \ we \ finally \ get \ f(C \cup B) = f(A \cup B \cup C) \ leading
to \ B \cap f(C \cup B) = \emptyset, \ that \ is \ C \succ B \ as \ desired.

Restricting the relation \( \succ \) thus defined to the elementary set \( \Omega \), we see
that the modular order induced on \( \Omega \) by a rational choice function \( f \) is simply
given by:

\[ x \succ y \text{ iff } y \notin f\{x, y\} \] that \( x \neq y \) and \( f\{x, y\} = \{x\}. \)

In other words, \( x \) is preferred to \( y \) iff \( x \) is the (unique) best element in the
set \( \{x, y\} \).

There exists a converse of theorem 5:
Theorem 6 Let $\succ$ be a rational preference order on $S = \wp(\Omega)$ and $f$ the function on $\wp(\Omega)$ that associates with any subset $X$ of $\Omega$ the subset of its $\succ$-maximal elements. Then $f$ is a rational choice function, and the order induced by $f$ is equal to $\succ$.

Proof: As usual, we denote by the same symbol $\succ$ the order on $\wp(\Omega)$ and the induced order on $\Omega$. Clearly the function $f$ defined as in the theorem satisfies Inclusion and Consistency. To show Arrow, let $X$ be a subset of $Y$ such that $X \cap f(Y) \neq \emptyset$. There exists an element $x$ of $X$ that is maximal among the elements of $Y$. Let $z \in f(X)$. We have the $z \in f(Y)$: otherwise $z$ would not be maximal in $Y$ and there would exist an element $y \in Y$ such that $y \succ z$. But, by the choice of $x$, we have $y \preceq x$, and by modularity this would imply $x \succ z$, a contradiction since $z$ is maximal in $X$. This shows that $f(X) \subseteq X \cap f(Y)$. The converse inclusion is trivial. It remains to show that the order $\succ_f$ induced by $f$ is equal to $\succ$. But if $x$ and $y$ are two different elements of $\Omega$, we have $x \succ_f y$ iff $f\{x, y\} = \{x\}$ and the latter equality holds iff $x \succ y$. The restriction to $\Omega$ of $\succ_f$ therefore agrees with that of $\succ$. As the preference rational orders $\succ_f$ and $\succ$ are totally determined by their restrictions to $\Omega$, we see that these orders are the same.

It follows from the above results that any choice function may be considered as the operation of extracting from a menu $X$ its best elements. The tight link we put in evidence between rational preference orders and rational choice functions suggests the existence of an analogue to the chain representation of rational preferences in the framework of rational choice functions. This is indeed the case as we shall see now.

3.2 Chain representation of rational functions

Let $\Delta$ be an $\Omega$-chain of length $h$, that is a chain of subsets of $\Omega$

$$D_0 \subset D_1 \ldots \subset D_{h-1}$$

with last term $D_{h-1} = \Omega$. Given such a sequence and an arbitrary subset $X$ of $\Omega$, recall that we denote by $r(X)$ the first index $i$ such that $X \cap D_i \neq \emptyset$; for any element $x \in \Omega$, we set $r(x) = r\{x\}$. We can associate with $\Delta$ the function $f$ on $\wp(\Omega)$ defined by

$$\forall X, f(X) = X \cap D_{r(X)}$$
It is immediate that this function is a rational choice function. Conversely, as expected from the duality between rational preference orders and rational choice functions, it is easy to show that any rational function is generated by a chain of subsets of $\Omega$:

**Theorem 7** Any rational function is induced by a $\Omega$-chain.

**Proof:** Rather than proving this result by considering the preference order associated with the rational function $f$ and its induced chain, we directly define inductively the chain $(D_i)$ by:

$$D_0 = f(\Omega) \quad \text{and} \quad D_{i+1} = D_i \cup f(\Omega - D_i)$$

Note that $(D_i)$ is a strictly increasing sequence: indeed, by inclusion and consistency, we have $\emptyset \neq f(\Omega - D_i) \subseteq (\Omega - D_i)$ for all $i < h - 1$, where $h$ is the length of the chain. We have to show that for every subset $X$ of $\Omega$ $f(X) = X \cap D_r(X)$. If $r(X) = 0$, we have $X \cap f(\emptyset) \neq \emptyset$, therefore, by Arrow condition, $f(X) = X \cap f(\Omega) = X \cap D_r(X)$. If $r(X) > 0$, we have $X \subseteq (\Omega - D_{r(X)-1})$ and $\emptyset \neq X \cap D_{r(X)} = X \cap f(\Omega - D_{r(X)-1})$. By Arrow condition again, this implies $f(X) = X \cap f(\Omega - D_{r(X)-1}) = X \cap D_r(X)$ as desired.

Note that the chain $\Delta$ thus defined is exactly the chain associated with the preference order $\succ$ induced by $f$: we have indeed, for any subsets $A$ and $B$ of $\Omega$ the following equivalences:

1. $A \succ f B$
2. $B \cap f(A \cup B) = \emptyset$
3. $B \cap ((A \cup B) \cap D_{r(A \cup B)}) = \emptyset$
4. $B \cap D_{r(A \cup B)} = \emptyset$
5. $D_{r(A \cup B)}$ meets $A$ and not $B$
6. $A \succ B$.

**Remark 5** The above result may be extended to infinite sets if $f$ is required to satisfy, instead of Consistency, the weaker condition $\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
3.3 Revising rational choice functions

The problem of revising a choice function arises when it appears necessary to modify its value on one or several subsets of its domain. This problem is clearly linked with that of revising a complete set of preferences: we saw indeed that, given a rational choice function on a set \( \Omega \), there exists a preference rational order among the menus of \( \Omega \), and the choice function just consists in establishing, for any menu \( X \) the list of the best elements of \( X \), that is those that are maximal for this preference relation. Revision therefore occurs when an element \( x \) of \( X \) that was considered as a best choice does not deserve anymore this qualification and has to be withdrawn from \( f(X) \), or, on the contrary when an element \( y \) has to be added to the initial list \( f(X) \). We will call contraction the first operation and revision the second one; conforming to the usual abuse of language, we will nevertheless refer to one or the other of these operations as a revision.

The representation of rational choice functions by subsets chains will be naturally our main tool to study the problem of revision. Nevertheless, contrary to what happened in the case of complete preferences sets, this representation is not sufficient by itself to provide a unique solution: suppose for instance we are in the simple situation where the function \( f \) is given by a chain of length 3, \( D_0 \subset D_1 \subset \Omega \), and an element \( x \) has to be removed from the list of the best elements of a set \( X \) such that \( r(X) = 1 \). Consider the following transformations of the original chain:

\[
\begin{align*}
D_0 & \subset D_1 - \{x\} \subset \Omega \\
D_0 & \subset D_1 - \{x\} \subset D_1 \subset \Omega \\
D_0 & \subset D_0 \cup D_1 \cap (X - \{x\}) \subset D_1 \subset \Omega.
\end{align*}
\]

Then any of these modifications gives raise to a choice function \( f' \) fairly close to \( f \) that satisfies Arrow and for which one has \( x \not\in f'(X) \), as desired. How can we determine our choice ? Even the constraints of the elementary principle of success (\( x \) should not be an element of \( f'(X) \)), of minimal change (the new set of the best elements of \( X \) should be \( f(X) - \{x\} \)), and of relative independence (if \( Y \cap X = \emptyset \), \( f'(Y) = f(Y) \)), all properties satisfied by the three modified chains, do not permit by itself to directly choose an optimal solution.

We will make use of the following

**Lemma 3** Let \( X \) be a subset of \( \Omega \) and \( x \) a element of \( X \). One has \( r(x) = r(X) \) iff \( x \in f(X) \).
Proof:
Suppose that \( x \) is an element of \( f(X) \). Then \( x \in f(X) = X \cap D_{r(X)} \), hence \( x \in D_{r(X)} \), and we see that \( r(x) \leq r(X) \). Since \( x \in X \cap D_{r(x)} \), this latter set is not empty, and we have therefore \( r(X) \leq r(x) \), hence the equality. The converse is immediate.

3.3.1 Chain contractions of choice functions

We want to withdraw an element \( x \) from the set \( f(X) \). This amounts to defining a new rational choice function \( f_{\perp(x)} \) such that \( x \notin f_{\perp(x)}(X) \). We have to suppose that \( f(X) \neq \{x\} \), since, otherwise, we would have \( f_{\perp(x)}(X) = \emptyset \) by the minimal change principle. Note that this condition is not anymore necessary if we deal with choice functions that satisfy the condition stated in remark 5.

Our guiding line will be the following: if \( x \) has to be retracted from the set of best elements of \( X \), this set being otherwise unchanged, this means that \( x \) does not match anymore the standards of the other elements of \( f(X) \), and consequently that any element \( y \neq x \) of \( f(X) \) will be now considered as better than \( x \). Thus if \( f' \) is the new choice function, we will have \( f'(x,y) = \{y\} \ \forall y \in f(X), \ y \neq x \). Note that this is equivalent to the condition \( f'(x,y) = \{y\} \) for some \( y \in f(X) \), since, by the principle of minimal change, \( f'(X) \) is bound to be equal to \( f(X) - \{x\} \). It follows that the inequality \( y \succ x \) has to be induced by the revised choice function \( f' \), and we must therefore look for an expansion of the set of preferences induced by the cain \( \Delta \) associated with \( f \). Observe now that, by the principle of minimal change again, we should preserve all the inequalities induced by \( f \): if, given arbitrary \( z \) and \( u \), one had \( \{z\} = f\{u,z\} \), so that \( z \) was considered as a strictly better choice than \( u \), depreciating \( x \) with respect to the elements of \( f(X) \) should not change this fact. In other words, all the preferences \( z \succ u \) should be maintained. We see therefore that the principle of minimal change in the framework of rational choice functions revision agrees with this same principle as studied in the framework of rational preferences. This justifies the application of the results established in the first part of section (2.3.3).

We let therefore \( y \) be an arbitrary element \( y \in f(X) - \{x\} \) and consider the auxiliary chain \( \Delta \) associated with \( f \). By what precedes, we have \( r(x) = r(y) \) and we are looking for an expansion of \( \Delta \) by \( y \succ x \). As established in the first part of this paper, such an expansion is induced by the
chain:

\[ D_0 \subset D_1 \subset \ldots \subset D_{r(x)-1} \subset D_{r(x)} \subset \{x\} \subset D_{r(x)} \ldots \subset D_{h-1}. \]

The revised choice function \( f' \) is therefore given on all sets \( Y \) by:

\[
f'(Y) = \begin{cases} 
  f(Y) & \text{if } x \not\in f(Y) \text{ or if } f(Y) = \{x\} \\
  f(Y) - \{x\} & \text{otherwise}
\end{cases}
\]

Note that the set \( X \) does not play any role in this contraction process.
This comes from the fact that the whole operation amounted to attributing an intermediate greater rank to all the elements \( t \neq x \) that were initially as good as \( x \) or, equivalently, an intermediate lower grade to \( x \). For instance, we can suppose that we had \( h - 1 = 20 \) and \( \kappa(x) = 12 \). Then changing \( f \) to \( f' \) is equivalent to raise to 12,5 the rank of all elements \( t \neq x \) that had rank 12, or to keep all initial ranks except for \( x \) that is retrograded to 11.5.

We close this paragraph with a remark: the process of contraction we described rests on the fact that we do not know why the element \( x \) has to be retracted from the set \( f(X) \): it is this very absence of information that commands the solution we proposed: indeed, if, for instance, one knows from the beginning that the element \( x \), that had initial rank \( \kappa(x) \), has to be retrograded to a given lower rank \( \kappa(x) - i \), the revision problem boils down to a much simpler one, an evident solution of which is for instance given by the chain

\[ \ldots \subset D_{r(X)-1} \subset D_{r(X)} - \{x\} \subset \ldots \subset D_{r(X)+i-1} - \{x\} \subset D_{r(X)+i-1} \subset \ldots \]

### 3.3.2 Chain expansions of choice functions

The problem is now to add a new element \( x \) to a list \( f(X) \). The minimal change principle requires this operation be done without unnecessarily adding \( x \) to any other list \( f(Y) \). It is clear, though, that this principle cannot be applied to the subsets \( Y \) of \( X \) such that \( x \in Y \), since, by Arrow condition, we will have then \( x \in f'(Y) \). Again, the solution we are now aiming at should be weighed by the fact we only have a default information and do not know on what grounds \( x \) should be added to the list \( f(X) \); we take for
granted that it was decided to reevaluate the original rank of \(x\), bringing it at the level \(\kappa(X)\), rather than because the elements of the original list were retrograded to \(\kappa(x)\).

With this in mind, we note that, since \(x \not\in f(X)\), one has \(z \succ x\) for all elements \(z \in f(X)\); after revision none of these inequalities must hold anymore, since both \(x\) and \(z\) are to be in \(f'(X)\). Thus the operation to perform implies a contraction by \(z \succ x\) for all \(z \in f'(X)\). But, by the minimal change principle, we should have \(f'(X) = f(X) \cup \{x\}\), and this implies that for all elements \(t \neq x\) of \(X - f(X)\) and all \(z \in f(X)\), the inequalities \(z \succ t\) have to be preserved. Therefore we have to perform a contraction with constraints, a problem that goes beyond the simple study we did in Part 1. The reader may check that a simple chain contraction as in (2.3.2) is quite inefficient. It would indeed yield the chain

\[ D_0 \subset D_1 \subset \ldots \subset D_{r(X)-1} \subset D_{r(x)} \subset D_{r(x)+1} \subset \ldots \subset D_{h-1}, \]

and this result is clearly unacceptable: it amounts to pushing back to \(\kappa(x)\) all the elements \(y\) of \(\Omega\) such that \(\kappa(x) \leq \kappa(y) \leq \kappa(X)\).

To perform the desired expansion, we cannot therefore use the results established in the framework of rational preferences. A direct study nevertheless provides an immediate and simple solution. Recall indeed that all we want is that \(x\) be as good as the best elements of \(f(X)\), the rank of these elements being unchanged. Thus we must have \(r'(x) = r(X)\), so \(x\) must be an element of \(W_{r(X)}\) and therefore an element of all the subsequent links of the chain. This leads to the chain

\[ \ldots \subset D_{r(X)-1} \subset D_{r(X)} \cup \{x\} \subset D_{r(X)+1} \cup \{x\} \subset \ldots \subset D_{r(x)} \subset \ldots \subset D_{h-1}. \]

We have \(f'(X) = f(X)\) as desired and \(g'(z) = g(z)\) for all elements \(z \neq x\). More precisely, the new choice function \(f'\) is given by:

\[ f'(Y) = \begin{cases} 
  f(Y) & \text{if } x \not\in Y \text{ or } r(Y) < r(X) \\
  f(Y) \cup \{x\} & \text{if } x \in Y \text{ and } r(Y) = r(X) \\
  \{x\} & \text{if } x \in Y \text{ and } r(Y) > r(X) 
\end{cases} \]

### 3.3.3 The revision of choice functions

Our last task will be now that of replacing an item of \(f(X)\) by another one. We call this a revision, but clearly this problem presents quite a difference
with both classical revision and the preference revision that was studied in
the first part of this paper. Indeed, these two latter domains treated the
same problem, which occurred when it was impossible to incorporate a new
information to a set of data (a knowledge base or a complete set of rational
preferences) without retracting at the same time some item that was not
coherent with the new piece of information. In the framework of choice
function where we are now working, the problem is different: as we saw,
it is always possible - we even dispose of several methods - to perform an
expansion, that is to decide that such or such element of a set \( X \) should be
added to \( f(X) \). We do not therefore have to face a dilemma, debating which
item should first be withdrawn from the list. Our problem reduces to the
following one: replace the element \( x \) of \( f(X) \) by the element \( x' \) of \( X - f(X) \).
To do so, we will naturally make use of the contraction and expansion process
deﬁned in the preceding paragraphs. Note that we have
\[ r(X) = r(x) < r(x') \]
by the choice of \( x \) and \( x' \). The original chain is
\[ D_0 \subset D_1 \subset \ldots \subset D_{r(x)} \subset \ldots \subset D_{r(x')} \subset \ldots \subset D_{h-1}. \]
The contraction by \( x \) first leads to the chain
\[ D_0 \subset D_1 \subset \ldots \subset D_{r(x)-1} \subset D_{r(x)} - \{x\} \subset D_{r(x)} \ldots \subset D_{r(x')} \subset \ldots \subset D_{h-1}. \]
For the subsequent expansion by \( x' \), we note that we have to distinguish
two cases, depending on whether \( f(X) \) boils down to \( \{x\} \) or not. In the first
case, we get as resulting chain
\[ \ldots \subset D_{r(x)-1} \subset D_{r(x)} - \{x\} \subset D_{r(x)} \cup \{x'\} \subset D_{r(x)+1} \cup \{x'\} \ldots \subset D_{r(x')} \subset \ldots \]
In the general case where \( f(X) \neq \{x\} \), the resulting chain is
\[ \ldots \subset D_{r(x)-1} \subset D_{r(x)} - \{x\} \cup \{x'\} \subset D_{r(x)} \cup \{x'\} \ldots \subset D_{r(x')} \subset \ldots \subset D_{h-1}. \]
As a matter of facts, this latter chain is exactly the one we obtain if,
instead of first contracting by \( \{x\} \) and then expand by \( \{x'\} \), one chooses first
to expand by \( \{x'\} \) and then to contract by \( \{x\} \). Thus, the two operations of
expansion and contraction deﬁned in the preceding paragraph commute and
give raise to a revision process in the principal case where \( f(X) \neq \{x\} \).
In both cases we have \( f'(X) = f(X) - \{x\} \cup \{x'\} \). Note that \( f'(Y) = f(Y) \)
for all subsets \( Y \)'s such that \( \kappa(Y) > \kappa(x) \) or \( \kappa(Y) \leq \kappa(x') \) or \( \{x, x'\} \cap Y = \emptyset \).
4 Conclusion

The tool provided by logical or subsets chains reveal quite a performing one in the study, be it static or dynamic, of the complete preference sets or the choice functions that describe an agent behavior in a finite environment. It may be probably carried over to arbitrary uncomplete sets of preference since we dispose of several methods to complete such sets. But it is probably not possible to extend this representation in the infinite case, as there exist rational preferences that cannot be represented by logical chains. Nevertheless it would be interesting to characterize in this framework the complete preference sets that can be so retrieved.

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